Announcements

1) Course Evaluations!

2) Survey – online soon, fill out for 2 extra credit points

3) HW 6 due Thursday

Step 2: Inverse Iteration

Start with μ a guess for an eigenvalue of A.

Like last time, choose $v^{(0)}$ to be an arbitrary vector of unit length.

Goal: approximate eigenvalues other than the eigenvalue of largest absolute value.

Algorithm. initialize at $v^{(0)}, \mu$.

For $i = 1, 2, 3, \ldots$

Solve $(A - \mu I)w = v^{(i-1)}$

Set $v^{(i)} = w / \|w\|_2$

Set $\lambda^{(i)} = (v^{(i)})^t A v^{(i)}$

Theorem: (rate of convergence)

Let the eigenvalues of A be $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$. Let λ_j be the closest eigenvalue of A to μ, and let λ_k be the second-closest, in the sense that

$$\boxed{|\mu - \lambda_j| < |\mu - \lambda_k| \leq |\mu - \lambda_i|}$$

$\forall\ 1 \leq i \leq m,\ i \neq j$.

Let $\{q_1, q_2, \dots, q_m\}$ be an orthonormal basis of eigenvectors of A corresponding to $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, respectively.

Remember that A is real and symmetric, so such a basis exists!

If $q_j^t v^{(0)} \neq 0$, then

$$\boxed{\| v^{(i)} - (\pm) q_j \|_2 = O\left(\frac{|u - \lambda_k|^i}{|u - \lambda_j|^i} \right)}$$

and

$$\boxed{| \lambda^{(i)} - \lambda_j | = O\left(\frac{|u - \lambda_k|^{2i}}{|u - \lambda_j|^{2i}} \right)}$$

Finally, Combine!

Rayleigh Quotient Iteration

Make an initial choice $v^{(0)}$ of unit length and an initial eigenvalue guess

$$\boxed{\lambda^{(0)} = \left(v^{(0)}\right)^t A v^{(0)}}$$

Algorithm: Initialize at vector $v^{(0)}$ and number $\lambda^{(0)}$.

For $i = 1, 2, 3, \ldots$

Solve $(A - \lambda^{(i-1)})w = v^{(i-1)}$

set $v^{(i)} = w / \|w\|_2$

set $\lambda^{(i)} = (v^{(i)})^t A v^{(i)}$

Example 1: (book)

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$v^{(0)} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\lambda^{(0)} = 5$$

At $\lambda^{(3)}$, we already have 15 decimal places of accuracy!

Book claims 2 more iterations increase the precision to 270 decimal places!

Theorem: (rate of convergence)

Given $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ eigenvalues of A and $\{q_1, q_2, \cdots, q_m\}$ the associated orthonormal basis of eigenvectors, then Rayleigh quotient iteration converges to a pair (λ_j, q_j) as follows:

If $v^{(0)}$ is close enough to q_j, then

$$\boxed{\|v^{(i+1)} - (\pm) q_j\|_2 = O\left(\|v^{(i)} - (\pm) q_j\|_2^3\right)}$$

and

$$\boxed{|\lambda^{(i+1)} - \lambda_j| = O\left(|\lambda^{(i)} - \lambda_j|^3\right)}$$

as $i \to \infty$.

Not every choice of $v^{(0)}$ produces an algorithm that converges! However, for a given A, the set of all vectors v that the algorithm does not converge for is Lebesgue measure zero.

Lebesgue measure is inherited from $\mathbb{R}$ on any vector space isomorphic to $\mathbb{R}^n$. Examples of Lebesgue measure zero sets in $\mathbb{R}$:

1) any finite set

2) any countable set ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}$)

3) The Cantor set (uncountable)

For a precise definition,

take Math 452/552!

The End!